HALE SCHOOL

Semester 2 Examination 2012

Question/answer booklet

MATHEMATICS: Specialist 3C/3D

Section One (calculator-free)

ANSWERS

Student's name

Circle teacher's initials MAV STL

Time allowed for this section

Reading time before commencing work: 5 minutes

Working time for section: 50 minutes

Material required/recommended for this section

To be provided by the supervisor

This Question/answer booklet for Section One, and a separate formula sheet which may also be used for Section Two

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this examination

	Number of questions available	Number of questions to be attempted	Working time (minutes)	Marks available
Section One Calculator—free	**7**	**7**	**50**	**50**
Section Two Calculator—assumed	11	11	100	100
			Total marks	150

Instructions to candidates

1. Answer all the questions in the spaces provided.

2. Spare answer pages are provided at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued i.e. give the page number.

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Correct answers given without supporting reasoning may not be allocated full marks. Incorrect answers given without supporting reasoning cannot be allocated any marks. If you repeat an answer to any question, ensure that you cancel the answers you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

Question 1 **(7 marks)**

Determine the following integrals:

(a) $\int \frac{\sin 2x}{4+3\cos^2 x}\,dx$ (3 marks)

$$\int \frac{\sin 2x}{4+3\cos^2 x}\,dx$$
$$= \int \frac{2\sin x\cos x}{4+3\cos^2 x}\,dx$$
$$= -\frac{1}{3}\int \frac{-6\sin x\cos x}{4+3\cos^2 x}\,dx$$
$$= -\frac{1}{3}\ln(4+3\cos^2 x)+c$$

Marking criteria	
Recognises the numerator is related to the derivative of the denominator	✓
Multiplies the integral by -1/3	✓
Integrates correctly, including c	✓

(b) $\int \cos^3 x\sin^2 x\,dx$ (Let $u = \sin x$) (4 marks)

$$\int \cos^3 x\sin^2 x\,dx$$
$$= \int \cos^2 x\sin^2 x\cos x\,dx \qquad \text{let } u=\sin x$$
$$= \int (1-u^2)u^2\,du$$
$$= \frac{u^3}{3}-\frac{u^5}{5}+c$$
$$= \frac{\sin^3 x}{3}-\frac{\sin^5 x}{5}+c$$

Marking criteria	
Uses correct substitution	✓
Simplifies $\cos^3 x\,dx$ in terms of u	✓
Integrates correctly, including c	✓
States the final answer in terms of x	✓

Question 2 **(4 marks)**

A function f(x) has the following properties:

$f(x) > 0$, $f(1) = 4$ and $f'(1) = 2$

(a) If $g(x) = \ln(f(x))$, find $g'(1)$. (2 marks)

$g(x) = \ln(f(x))$

$\frac{dg}{dx} = \frac{1}{f(x)} f'(x)$

$\therefore \quad g'(1) = \frac{1}{4}(2) = \frac{1}{2}$

Differentiates ln function ✔
Substitutes values to determine g'(1) ✔

(b) If $h(x) = f(\sqrt{x})$, find $h'(1)$. (2 marks)

$h(x) = f(\sqrt{x})$

$\frac{dh}{dx} = \frac{1}{2\sqrt{x}} f'(\sqrt{x})$

$\therefore \quad h'(1) = \frac{1}{2\sqrt{1}}(2) = 1$

Uses chain rule correctly ✔
Substitutes values to determine h'(1) ✔

Question 3 **(7 marks)**

(a) Prove the following result: $\lim_{x\to 0}\frac{1-\cos x}{x}=0$ (4 marks)

$$\lim_{x\to 0}\frac{1-\cos x}{x}$$
$$=\lim_{x\to 0}\frac{1-\cos x}{x}\times\frac{1+\cos x}{1+\cos x}$$
$$=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$
$$=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}=\lim_{x\to 0}\frac{\sin x}{x}\times\frac{\sin x}{1+\cos x}$$
$$=1\times\frac{0}{2}$$
$$=0$$

Multiplies the limit by (1+cos x)/(1+cos x)	✓
Simplifies numerator to $\sin^2 x$	✓
Expresses the limit as the limit of two factors	✓
Uses the result (sin x)/x → 1 as x → 0	✓

(b) Evaluate the following limit: $\lim_{x\to \pi}\frac{\sin\frac{1}{2}(\pi-x)}{x-\pi}$ (3 marks)

$$\lim_{x\to \pi}\frac{\sin\frac{1}{2}(\pi-x)}{x-\pi}$$
$$=-\lim_{y\to 0}\frac{\sin\frac{1}{2}y}{y}$$
$$=-\frac{1}{2}\lim_{y/2\to 0}\frac{\sin y/2}{y/2}$$
$$=-\frac{1}{2}$$

Substitutes $x-\pi$ by y, including the limit and the negative sign	✓
Changes y → y/2, including the factor 1/2	✓
States the answer	✓

Question 4 **(11 marks)**

The points P and Q have position vectors given by **i** + 2**j** – 4**k** and 3**i** – **j** + 2**k** respectively. The line joining PQ cuts the x-z plane at R.

(a) Find the position vector of the point R. (5 marks)

$$\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$$

$$\text{Eq of the line PQ}: \quad \vec{r} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$$

$$\text{cuts the } x-z \text{ plane} \rightarrow y = 0$$

$$\therefore \quad 2 - 3\lambda = 0 \quad \rightarrow \quad \lambda = \frac{2}{3}$$

$$\text{So} \quad \overrightarrow{OR} = \begin{pmatrix} 7/3 \\ 0 \\ 0 \end{pmatrix}$$

Determines the vector PQ	✓
Finds the vector equation of the line PQ	✓
Expresses y = 0 when the line cuts the x-z plane	✓
Solves for λ	✓
Substitutes λ into line equation to determine the position vector OR	✓

(b) Find the ratio at which the point R divides PQ. (3 marks)

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OP} + h\overrightarrow{PQ}$$

$$\Rightarrow \quad h = \lambda = \frac{2}{3}$$

So R divides PQ in the ratio 2 : 1

Expresses OR in terms of OP and PQ, including a constant	✓
Deduces the constant = λ	✓
States the answer	✓

(c) Find the vector equation of the plane that passes through R and is perpendicular to PQ. (3 marks)

$\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ is normal to the plane

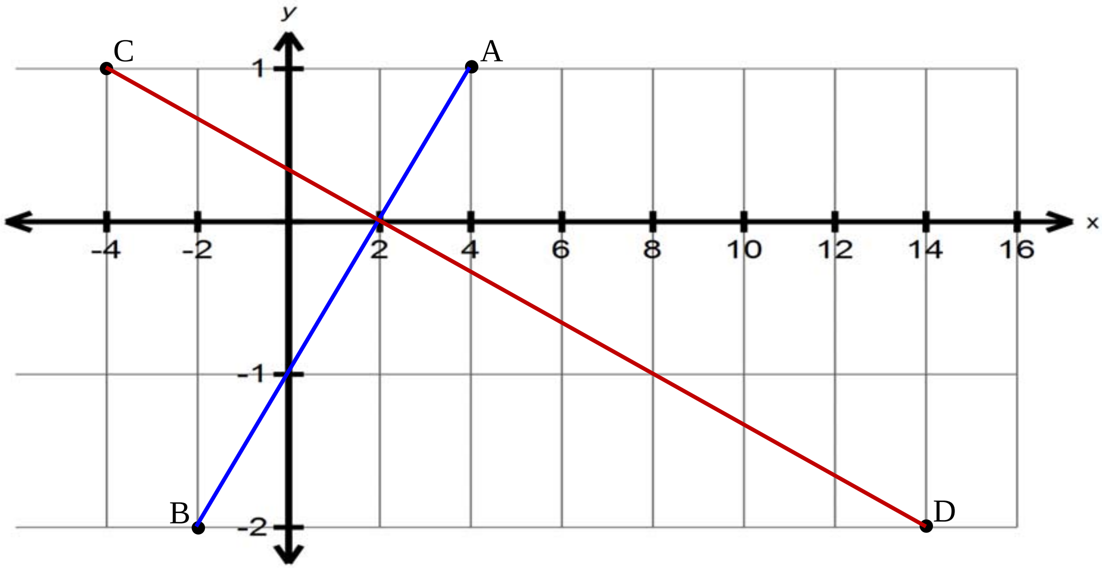
Eq of the plane : $\vec{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 7/3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

So $\vec{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = \dfrac{14}{3}$

States the normal vector to π ✓
Uses the coordinates of R to give the normal form of π ✓
Carries out the dot product ✓

Question 5 **(8 marks)**

A line segment AB is transformed by a matrix T **four** times to become CD as shown below.



(a) The effect of the 4 transformations by T can be carried out by a single transformation matrix S. State the relationship between T and S. (1 mark)

$T^4 = S$

States the answer ✓

(b) Determine the matrix S. (3 marks)

$$S\begin{bmatrix}4 & -2\\1 & -2\end{bmatrix}=\begin{bmatrix}-4 & 14\\1 & -2\end{bmatrix}$$

$$S=\begin{bmatrix}-4 & 14\\1 & -2\end{bmatrix}\times\begin{bmatrix}4 & -2\\1 & -2\end{bmatrix}^{-1}$$

$$S=\frac{1}{-6}\begin{bmatrix}-4 & 14\\1 & -2\end{bmatrix}\times\begin{bmatrix}-2 & 2\\-1 & 4\end{bmatrix}$$

$$\therefore \quad S=\begin{bmatrix}1 & -8\\0 & 1\end{bmatrix}$$

Sets up correct matrix equation ✓

Post-multiplies the above equation by the inverse of the coordinates matrix ✓

Simplifies correctly ✓

(c) Describe the geometric effect of matrix S. (2 marks)

Shear parallel to the x – axis, scale factor – 8.

Describe shearing parallel to the x-axis ✓
Specifies the scale factor ✓

(d) Determine the matrix T. (2 marks)

Let $T = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ then $T^4 = \begin{bmatrix} 1 & 4k \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 4k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ 0 & 1 \end{bmatrix} \Rightarrow k = -2$$

$$\therefore \quad T = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

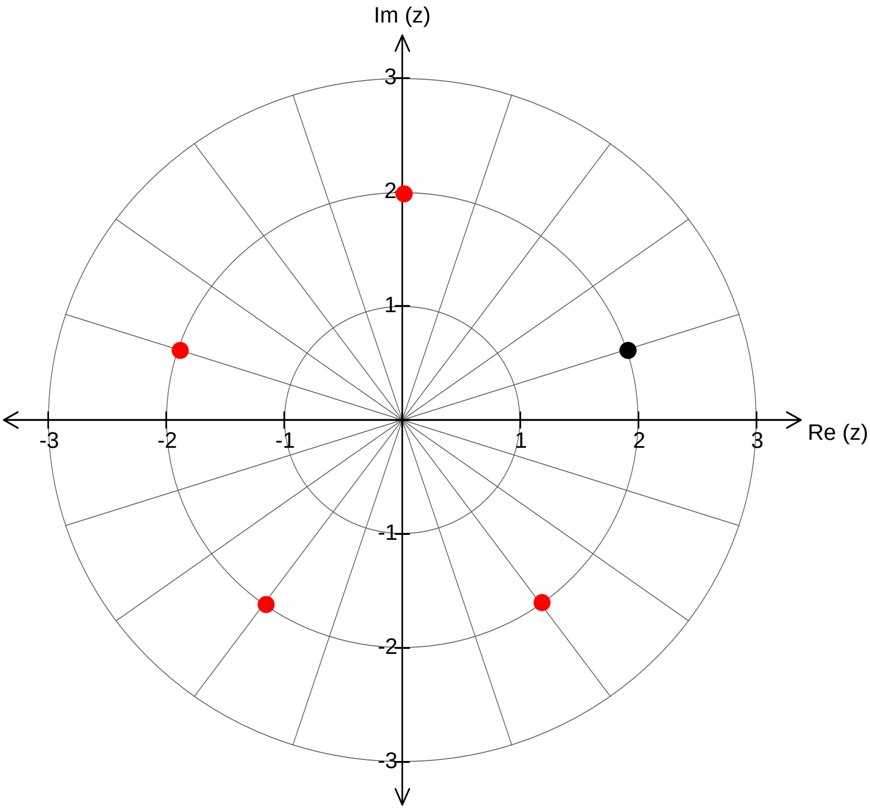
Represents T and T^4 by horizontal shear matrix ✓
Solves for k ✓

Question 6 **(7 marks)**

The complex plane below shows **one** of the roots of the following equation:

$$z^5 = u$$

where u is a complex number.



(a) Locate clearly, on the complex plane above, all the other roots of the equation

$$z^5 = u$$

Shows correct angle between roots ✓
Shows same modulus for each root ✓

(b) Determine the complex number u. (2 marks)

$$u = (2 \text{ cis} \frac{\pi}{10})^5 = 32 \text{ cis} \frac{\pi}{2}$$
$$= 32\, i$$

Determines the given root in polar form ✓
Simplifies correctly ✓

(c) Determine the sum of **all** the roots of the equation $z^5 = u$. Show your working/reasoning clearly. (3 marks)

A rotation of $\frac{2\pi}{5}$ maps the regular pentagon onto itself.

So $\sum \text{roots} = \sum (e^{i\frac{2\pi}{5}} \times \text{roots}) = e^{i\frac{2\pi}{5}} \sum \text{roots}$

$\Rightarrow \sum \text{roots} = 0$

Provides a reasonable working or explanation:	
Recognises a rotation of $2\pi/5$ has no effect on the sum of roots	✓
Expresses the rotation as $e^{2\pi i/5} \times \text{root}$	✓
States the answer	✓

Question 7 **(6 marks)**

Prove by contradiction that for any two integers a and b: $a^2 - 4b \neq 2$.

Assume $a^2 - 4b = 2$

Then $a^2 = 4b + 2 = 2(2b+1)$

$\Rightarrow a^2$ is a multiple of 2

$\Rightarrow$ a is a multiple of 2

Let $a = 2c$

$4c^2 = 2(2b+1)$

$\Rightarrow \quad 2c^2 = 2b + 1$

$\Rightarrow \quad \text{even} = \text{odd}$

So the assumption was wrong

Therefore $a^2 - 4b \neq 2$

States the correct assumption	✓
Expresses a^2 as a multiple of 2	✓
Deduces a as a multiple of 2	✓
Uses suitable for a to simplify the expression	✓
Recognises even+odd = odd	✓
Draws valid conclusion	✓

Additional working space

Question number(s): ________________

HALE SCHOOL

Semester 2 Examination 2012
Question/answer booklet

MATHEMATICS: Specialist 3C/3D

Section Two
(calculator-assumed)

ANSWERS

Student's name

Circle teacher's initials MAV STL

Time allowed for this section

Reading time before commencing work: 10 minutes
Working time for section: 100 minutes

Material required/recommended for this section

To be provided by the supervisor

This Question/answer booklet for Section Two. Candidates may use the separate formula sheet from Section One.

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler
Special items: drawing instruments, templates, notes on up to two unfolded sheets of A4 paper, and up to three calculators, CAS, graphic or scientific, which satisfy the conditions set by the Curriculum Council for this course.

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Question 8 **(7 marks)**

A plane is flying with velocity **v** = 0.04**i** + 0.03**j** + 0.12**k** km/s where the unit vectors **i**, **j** and **k** point East, North and vertically upwards respectively.

The initial position of the plane is 20 km South of the airport, at a height of 1.0 km. Find:

(a) the speed of the plane, (1 mark)

$$|v| = \sqrt{0.04^2 + 0.03^2 + 0.12^2}$$
$$= 0.13 \text{ km/s}$$

States the correct answer ✓

(b) the angle (nearest degree) of ascent of the plane, (3 marks)

Project $\vec{v}$ on the x – y plane: $0.04\hat{i} + 0.03\hat{j}$

So the angle of ascent is given by

$$\cos\theta = \frac{(0.04, 0.03, 0.12)\cdot(0.04, 0.03, 0)}{0.13 \times \sqrt{0.04^2 + 0.03^2}}$$

$\therefore \quad \theta \approx 67°$

Finds 0.04**i** + 0.03**j** + 0**k** ✓

Uses cosine rule to find angle ✓

States the correct answer ✓

(c) the time (whole number) at which the plane is closest to the airport. (3 marks)

$$\vec{r}(t) = 0.04t\,\hat{i} + (0.03t - 20)\,\hat{j} + (0.12t + 1)\,\hat{k}$$
$$|\vec{r}(t)|^2 = (0.04t)^2 + (0.03t - 20)^2 + (0.12t + 1)^2$$
$$= 0.0169t^2 - 0.96t + 401$$
$$|\vec{r}(t)|_{\min} \quad \text{when} \quad t = -\frac{-0.96}{2 \times 0.0196}$$
$$\approx 25 \text{ s}$$

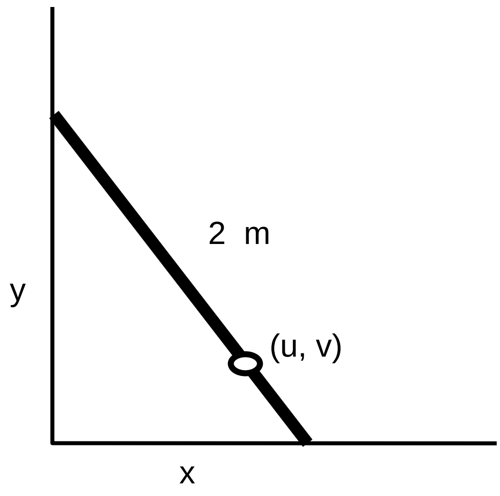
Determines r(t) ✓

Finds the distance in terms of t ✓

States the correct answer ✓

Question 9 **(12 marks)**

A ladder, 2 metres long, has its base on level ground and its top resting against a vertical wall. A ring is fixed 0.5 m from the base of the ladder as shown below. The ladder starts to slip down at a constant rate of 0.1 m/s when it is $\sqrt{3}$ metres up the wall.



(a) How fast (exact value) is the **foot** of the ladder moving away from the wall initially. (5 marks)

$$x^2 + y^2 = 2^2$$

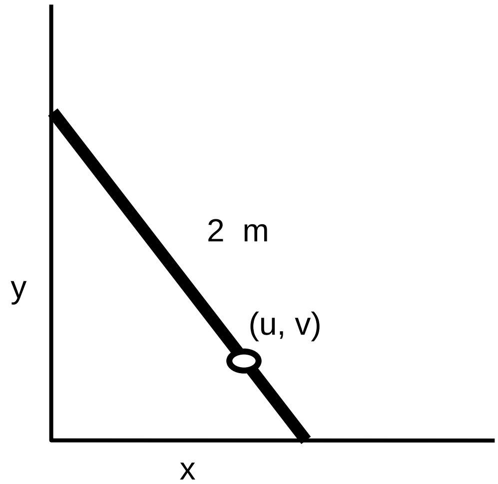
$$\Rightarrow \frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$$

$$= -\frac{\sqrt{3}}{1} \times (-0.1)$$

$$= \frac{\sqrt{3}}{10} \text{ m/s}$$

Sets up the equation relating x and y	✓
Expresses dx/dt in terms of dy/dt	✓
Determines the initial value of x	✓
Substitutes values	✓
States the answer	✓

(b) How fast is the ring moving down (vertically)? (3 marks)



Similar triangles :

$$\frac{y}{v} = \frac{4}{1} \Rightarrow \frac{dv}{dt} = \frac{1}{4}\frac{dy}{dt}$$

$$\therefore \frac{dv}{dt} = -\frac{1}{40}$$

Ring moving down at $\frac{1}{40}$ m/s

Sets up the equation relating v and y
Expresses dv/dt in terms of dy/dt
States the answer

(c) How far is the ladder up the wall when the **ring** is moving with a speed of $\frac{1}{20}$ m/s.

(4 marks)

$$\frac{du}{dt} = \frac{3}{4}\frac{dx}{dt} = \frac{3}{4}\left(\frac{y}{x} \times 0.1\right)$$

$$= \frac{3}{40} \times \frac{y}{\sqrt{4 - y^2}}$$

$$\text{speed}^2 = \left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2$$

$$\text{So} \quad \left(\frac{1}{20}\right)^2 = \left(\frac{3}{40} \times \frac{y}{\sqrt{4 - y^2}}\right)^2 + \left(\frac{1}{40}\right)^2$$

$$\Rightarrow \quad y = 1 \text{ m} \qquad \text{(ignore negative)}$$

Expresses du/dt in terms of dx/dt
Expresses du/dt in terms of y only
Sets up equations relating speed and velocity components
Solves equation

Question 10 **(9 marks)**

A particle moves along a straight line and its displacement x metres from a fixed point O on the line after t seconds is given by:

$$x = 10 \sin (kt - \theta)$$

where k and θ are positive constants and $0 \le \theta < \pi/2$.

The particle passes through the point O for the first time after 2 seconds and for the second time after 7 seconds.

(a) Find the values of k and θ. (4 marks)

$$\frac{1}{2}(\text{period}) = \frac{1}{2}\left(\frac{2\pi}{k}\right) = 7 - 2$$

$$\therefore \quad k = \frac{\pi}{5}$$

$$t = 2, \; x = 0 \quad \Rightarrow \quad \frac{\pi}{5}(2) - \theta = 0$$

$$\therefore \quad \theta = \frac{2\pi}{5}$$

Sets up the equation for ½ period ✓

Solves for k ✓

Uses $x = 0$ when $t = 2$ ✓

Solves for θ ✓

The particle is furthest away from O the second time when $t = T$ seconds.

(b) (i) Determine the value of T. (2 marks)

$$\text{second time} \Rightarrow \frac{\pi}{5}T - \frac{2\pi}{5} = \frac{3\pi}{2}$$

$$\therefore \quad T = 9.5 \text{ s}$$

Shows 2nd max displacement occurs when $kT - \theta = 3\pi/2$ ✓

Solves for T ✓

(ii) Find the distance travelled by the particle for the first T seconds. (3 marks)

$$x(0) = -10\sin\left(\frac{2\pi}{5}\right)$$

$$\therefore \quad \text{distance} = 30 + 10\sin\left(\frac{2\pi}{5}\right)$$

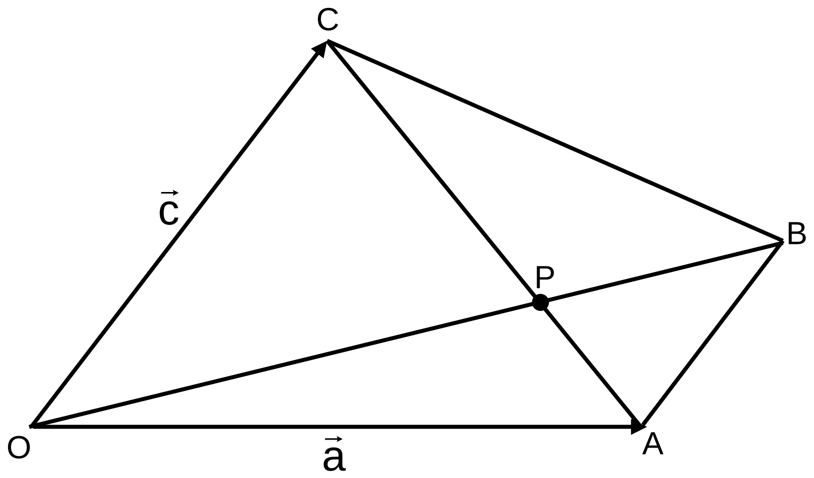
$$= 39.51 \text{ m}$$

Marking criteria	
Determines x(0)	✓
Shows distance given by | x(0) | + 30	✓
States the answer	✓

Question 11 **(7 marks)**

In the figure below, OABC is a trapezium with AB parallel to OC and 2AB = OC. The diagonals intersect at P.

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OC} = \vec{c}$



(a) Express $\overrightarrow{AC}$ and $\overrightarrow{OB}$ in terms of $\vec{a}$ and $\vec{c}$ (2 marks)

$$\overrightarrow{AC} = \vec{c} - \vec{a}$$
$$\overrightarrow{OB} = \vec{a} + \frac{1}{2}\vec{c}$$

States the answers ✓✓

Let $\overrightarrow{AP} = \lambda\,\overrightarrow{AC}$

(b) Express $\overrightarrow{OP}$ in terms of λ, $\vec{a}$ and $\vec{c}$. (2 marks)

$$\overrightarrow{OP} = \vec{a} + \overrightarrow{AP}$$
$$= \vec{a} + \lambda(\vec{c} - \vec{a})$$
$$= (1 - \lambda)\vec{a} + \lambda\vec{c}$$

Expresses **OP** as **OA** + λ**AC** ✓

Simplifies correctly ✓

(c) Determine the value of λ. (3 marks)

$$\overrightarrow{OP} = (1-\lambda)\vec{a} + \lambda\vec{c}$$

Let $\overrightarrow{OP} = \mu\overrightarrow{OB}$

Then $\overrightarrow{OP} = \mu(\vec{a} + \frac{1}{2}\vec{c})$

$\Rightarrow \quad 1-\lambda = \mu \quad$ and $\quad \lambda = \frac{1}{2}\mu$

$\therefore \quad \lambda = \frac{1}{3}$

Expresses **OP** in terms of **OB** with a constant	✓
Equates corresponding components from (b) and (c)	✓
Solves for λ	✓

Question 12 **(12 marks)**

A population of female bats living in a cave is studied and the following data is collected.

Age (months)	0 – 6	6 – 12	12 – 18	18 – 24
Initial population	4500	1800	900	130
Birth rate	0	1.9	1.5	0.7
Survival rate	0.5	0.8	0.4	0

The initial female population is represented by a column matrix as shown below.

$$P_o = \begin{bmatrix} 4500 \\ 1800 \\ 900 \\ 130 \end{bmatrix} \begin{matrix} 0-6 \\ 6-12 \\ 12-18 \\ 18-24 \end{matrix}$$

(a) Use a Leslie matrix L to represent the above birth rates and survival rates so that it can be used to calculate the female populations for subsequent years. (1 mark)

$$L = \begin{bmatrix} 0 & 1.9 & 1.5 & 0.7 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$$

States the correct answer ✓

(b) Write down a matrix equation that can be used to find the female population for each age group of the bats after 6 months. Do not evaluate. (1 mark)

$$P_1 = L \times P_o$$

States the correct answer ✓

(c) What is the **total** female population after 4 years? (2 marks)

$$P_8 = L^8 \times P_o = \begin{bmatrix} 0 & 1.9 & 1.5 & 0.7 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}^8 \times \begin{bmatrix} 4500 \\ 1800 \\ 900 \\ 130 \end{bmatrix} = \begin{bmatrix} 22163 \\ 9013 \\ 5866 \\ 1903 \end{bmatrix}$$

After 4 years, total population $= [1 \quad 1 \quad 1 \quad 1] \times P_8 = 38945$

Uses L^8 to find P_8	✓
Gives the total population	✓

(d) Given that the total female population after 5 years is 58890, find the percentage increase in the population every 6 months from year 4 to year 5. (2 marks)

$$1 + \text{rate} = \sqrt{\frac{58890}{38945}} = 1.2297$$

$\therefore$ Population is increasing at a rate of 23% every 6 months.

Sets up the correct equation	✓
States the answer	✓

Culling (Harvesting) is carried out for the age group 6 – 12 months at a rate of 30% with the intention of maintaining a stable population. The culling rate affects both the birth rate and the survival rate.

(e) Write down the new Leslie matrix. (1 mark)

$$L' = \begin{bmatrix} 0 & 1.33 & 1.5 & 0.7 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.56 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$$

States the correct answer	✓

(f) If P_n is the stable population, write down the population P_{n+1} after 6 months in terms of P_n and/or some constants and matrices (if necessary). (1 mark)

$$P_{n+1} = P_n$$

States the answer ✓

(g) Determine whether or not 30% is a reasonable culling rate in order to maintain a stable population. (4 marks)

$$P_{12} = [1 \quad 1 \quad 1 \quad 1] \times (L')^{12} \times P_o = 15075$$

$$P_{14} = [1 \quad 1 \quad 1 \quad 1] \times (L')^{14} \times P_o = 17028$$

$$P_{16} = [1 \quad 1 \quad 1 \quad 1] \times (L')^{16} \times P_o = 19234$$

$\Rightarrow$ population is increasing

So 30% cannot maintain a stable population.

Determines the trend of population change (use at least 3 populations for some large t)	✓✓
Provides a reason from part (f)	✓
Draws a valid conclusion	✓

OR

$$\begin{bmatrix} 0 & 1.9\times(1-h) & 1.5 & 0.7 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.8\times(1-h) & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\begin{cases} 1.9\times(1-h)b + 1.5c + 0.7d = a \\ 0.5a = b \\ 0.8\times(1-h)b = c \\ 0.4c = d \end{cases} \quad \Rightarrow \quad h \approx 0.4$$

$\therefore$ not reasonable

Question 13 **(7 marks)**

Prove by mathematical induction that, if n is a positive integer,

$$n\cdot 1+(n-1)\cdot 2+(n-2)\cdot 3+......+2\cdot(n-1)+1\cdot n=\frac{1}{6}n(n+1)(n+2)$$

$P(n):\quad n\cdot 1+(n-1)\cdot 2+(n-2)\cdot 3+......+2\cdot(n-1)+1\cdot n=\frac{1}{6}n(n+1)(n+2)$

When $n=1$:

$LHS=1\cdot 1=1$ and $RHS=\frac{1}{6}(1)(1+1)(1+2)=1$

$\therefore$ $P(1)$ is true.

Assume $P(n)$ is true for $n=k$ where k is a positive integer.

Thus $k\cdot 1+(k-1)\cdot 2+(k-2)\cdot 3+......+2\cdot(k-1)+1\cdot k=\frac{1}{6}k(k+1)(k+2)$

Consider $n=k+1$:

$$\begin{aligned}
LHS&=(k+1)\cdot 1+(k)\cdot 2+(k-1)\cdot 3+......+2\cdot(k)+1\cdot(k+1)\\
&=(k\cdot 1+1)+(k-1+1)\cdot 2+(k-2+1)\cdot 3+......+(1\cdot k+k)+(k+1)\\
&=[\,1+2+......+(k-1)+k+(k+1)\,]\\
&\quad+[\,k\cdot 1+(k-1)\cdot 2+(k-2)\cdot 3+......+2\cdot(k-1)+1\cdot k\,]\\
&=[\,1+2+......+(k-1)+k+(k+1)\,]+\frac{1}{6}k(k+1)(k+2)\\
&=\frac{1+k+1}{2}(k+1)+\frac{1}{6}k(k+1)(k+2)\\
&=\frac{1}{6}(k+1)(k+2)(k+3)\\
&=RHS
\end{aligned}$$

$\therefore$ $P(k+1)$ is true if $P(k)$ is true.

By MI, $P(n)$ is true for all positive integer.

Marking criteria	
Uses proper proof structure, including from LHS to RHS	✓
Proves P(1) is true	✓
States the assumption for P(k)	✓
Considers P(k+1), including the expression for LHS	✓
Simplifies the LHS of P(k+1) in order to use the assumption result	✓
Finds the sum of the AP	✓
Draws valid conclusion	✓

Question 14 **(9 marks)**

(a) Determine the value of A if $\dfrac{200}{x(200-x)} = \dfrac{1}{x} + \dfrac{A}{200-x}$. (2 marks)

$$\frac{200}{x(200-x)} = \frac{1}{x} + \frac{A}{200-x}$$
$$= \frac{200-x+Ax}{x(200-x)}$$
$$\therefore \quad A = 1$$

Simplifies the RHS of the equation	✓
Equates the numerators to solve for A	✓

Let $P(t)$ be the population of a certain animal species. Assume that $P(t)$ satisfies the following equation:

$$\frac{dP}{dt} = 0.2P(1-\frac{P}{200}) \quad \text{and} \quad P(0) = 150$$

(b) (i) Find P in terms of t. (5 marks)

$$\frac{dP}{dt} = 0.2P(1-\frac{P}{200}) \Rightarrow \frac{200}{P(200-P)}dP = 0.2\,dt$$
$$\int(\frac{1}{P} + \frac{1}{200-P})\,dP = \int 0.2\,dt$$
$$\ln(\frac{P}{200-P}) = 0.2t + c$$
$$\Rightarrow \quad P = \frac{200\,C\,e^{0.2t}}{1+C\,e^{0.2t}}$$
$$t = 0,\ P = 150 \Rightarrow C = 3$$
$$\therefore \quad P = \frac{600\,e^{0.2t}}{1+3\,e^{0.2t}}$$

Separates the variables	✓
Uses result (a) to integrate	✓
Rearranges and simplifies terms	✓
Uses initial value to determine C	✓
States the answer	✓

(ii) What is the long term behaviour of the population P(t)? (2 marks)

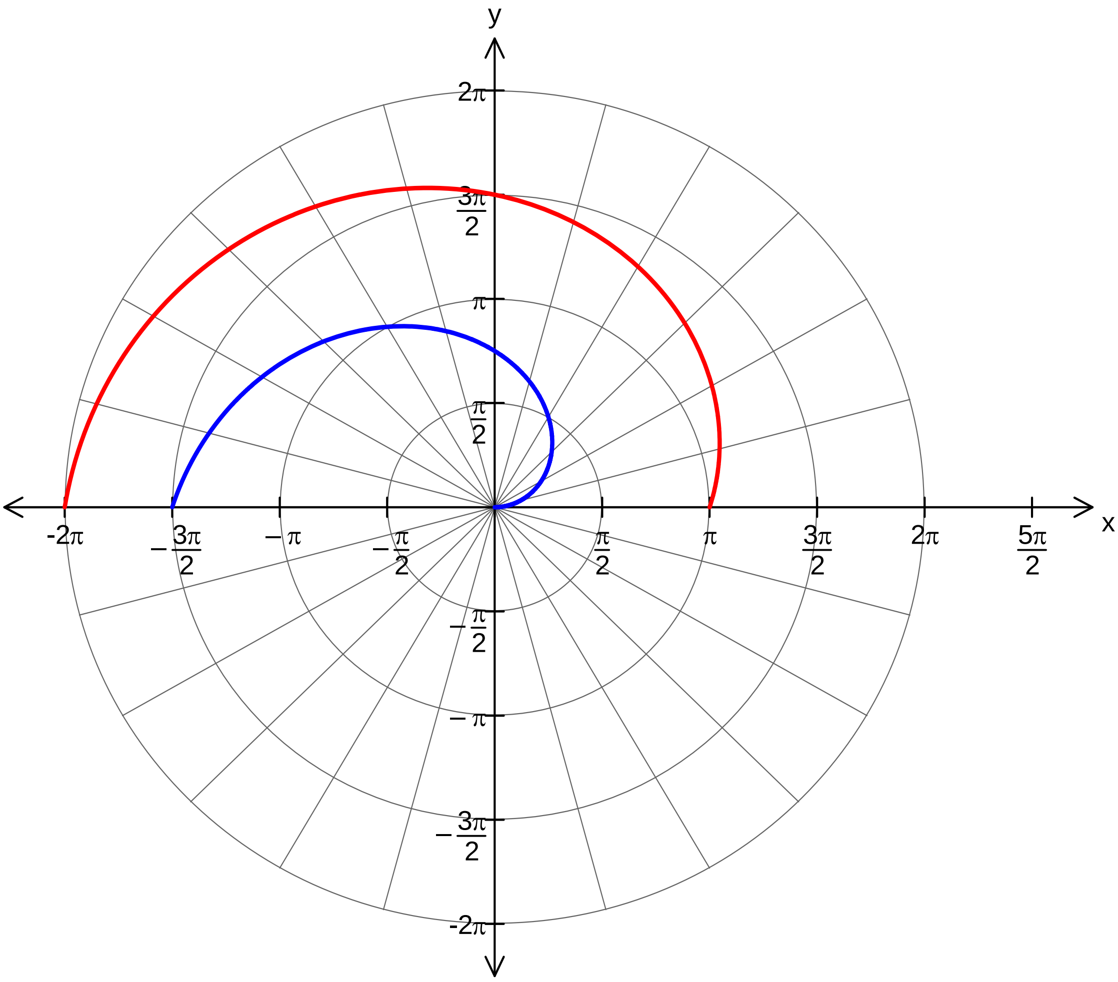
$$P = \frac{600\,e^{0.2t}}{1+3\,e^{0.2t}} \to \frac{600\,e^{0.2t}}{3\,e^{0.2t}} \quad \text{as} \quad t \to \infty$$

$$\therefore \quad P \to 200 \quad \text{as} \quad t \to \infty$$

Shows the dominant term in the denominator as t → ∞	✓
States the answer	✓

Question 15 **(10 marks)**

The diagram below shows the polar graphs of $r = k\theta$ and $r = \theta + c$ where k and c are constants and $0 \le \theta \le \pi$.



(a) Determine the values of k and c. (4 marks)

$$r = k\theta: \quad \frac{3\pi}{2} = k \times \pi \quad \Rightarrow \quad k = \frac{3}{2}$$

$$r = \theta + c: \quad \pi = 0 + c \quad \Rightarrow \quad c = \pi$$

Chooses a nice point on r = kθ to determine k	✓
Solves for k	✓
Chooses a nice point on r = θ + c to determine c	✓
Solves for c	✓

(b) Points $A(r, \alpha)$ and $B(r, \beta)$ are on the graphs of $r = k\theta$ and $r = \theta + c$ respectively such that they have the same r and the distance between them is $\sqrt{3}\,\pi$.

(i) Show that α satisfies $\frac{2}{3}\pi^2 = \alpha^2(1+\cos\frac{\alpha}{2})$. (4 marks)

$$r = \frac{3}{2}\alpha = \beta + \pi \qquad (0 \le \beta < \alpha \le \pi)$$

$$\Rightarrow \quad \beta = \frac{3}{2}\alpha - \pi$$

$$(\sqrt{3}\,\pi)^2 = r^2 + r^2 - 2r^2\cos(\alpha - \beta)$$

$$3\pi^2 = 2r^2(1 - \cos(\pi - \frac{\alpha}{2}))$$

$$3\pi^2 = 2(\frac{3}{2}\alpha)^2(1 + \cos\frac{\alpha}{2})$$

$$\therefore \quad \frac{2}{3}\pi^2 = \alpha^2(1 + \cos\frac{\alpha}{2})$$

Marking criteria	
Expresses β in terms of α	✓
Uses Cosine Rule for the distance AB	✓
Substitutes values into equation, including $\alpha - \beta$ in terms of α	✓
Simplifies correctly	✓

(ii) Determine the value(s) of α. (2 marks)

Solve $\frac{2}{3}\pi^2 = \alpha^2(1 + \cos\frac{\alpha}{2})$

From calculator :

$\alpha = 2.0944, \; 4.6975, \; ......$

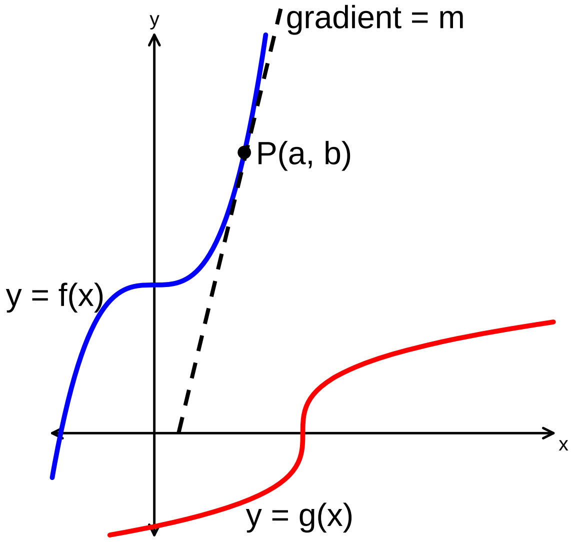
Choose $\alpha = 2.0944 \; (= \frac{2\pi}{3})$

Marking criteria	
Solves equation to obtain values of α	✓
Chooses the correct value of α	✓

Question 16 **(9 marks)**

The diagram below shows the graph of $y = f(x)$ and the graph of its inverse function $y = g(x) = f^{-1}(x)$.

A point $P(a, b)$ is on the graph of $y = f(x)$. The tangent at P has a gradient m.



(a) State the value of $g(f(a))$. (1 mark)

$g(f(a)) = a$

States the correct answer ✓

(b) Show that $g'(b) = 1/m$. (4 marks)

$$g(f(x)) = x$$
$$g'(f(x))f'(x) = 1$$
$$\Rightarrow \quad g'(f(x)) = \frac{1}{f'(x)}$$
$$g'(f(a)) = \frac{1}{f'(a)}$$
$$\therefore \quad g'(b) = \frac{1}{m}$$

Expresses the inverse relationship between f and g ✓
Uses chain rule to differentiate ✓
Simplifies correctly ✓
Substitutes values ✓

(c) Find the coordinates of the point of intersection of the tangent at P and the tangent at $x = b$ on the graph of $y = g(x)$ in terms of a, b and m. (Assume $m \neq -1$)

(4 marks)

tangent at (a, b) on $y = f(x)$: $y - b = m(x - a)$ ......(1)

tangent at (b, a) on $y = g(x)$: $y - a = \frac{1}{m}(x - b)$

$my - ma = x - b$ ......(2)

(1) and (2)

$\Rightarrow \quad x = y \qquad (\because \quad m \neq -1)$

$\therefore \quad x = \frac{ma - b}{m - 1}$

Coordinates: $\left(\frac{ma - b}{m - 1}, \frac{ma - b}{m - 1}\right)$

Determines the equation of tangent at (a, b) on f(x)	✓
Determines the equation of tangent at (b, a) on g(x)	✓
Shows x = y	✓
Determines coordinates	✓

Question 17 **(10 marks)**

(a) Evaluate $\int \frac{\tan^n x}{\cos^2 x}\,dx$ (in terms of n) where $n = 0, 1, 2, \ldots\ldots$ (2 marks)

$$\int \frac{\tan^n x}{\cos^2 x}\,dx$$
$$= \int \frac{1}{\cos^2 x}(\tan x)^n\,dx$$
$$= \frac{\tan^{n+1} x}{n+1} + c$$

Recognises $1/\cos^2 x$ is the derivative of $\tan x$ ✓
Integrates correctly, including c ✓

(b) If $F(n) = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$ where $n = 0, 1, 2, \ldots\ldots$, show that:

$$F(n+2) = \frac{1}{n+1} - F(n)$$

(5 marks)

$$F(n+2) = \int_0^{\pi/4} \tan^{n+2} x\,dx$$
$$= \int_0^{\pi/4} \tan^n x \tan^2 x\,dx$$
$$= \int_0^{\pi/4} \tan^n x \left(\frac{1}{\cos^2 x} - 1\right) dx$$
$$= \int_0^{\pi/4} \frac{\tan^n x}{\cos^2 x}\,dx - \int_0^{\pi/4} \tan^n x\,dx$$
$$= \frac{1}{n+1}[\tan^{n+1} x]_0^{\pi/4} - F(n)$$
$$= \frac{1}{n+1} - F(n)$$

Considers $\tan^{n+2}x$ as $(\tan^n x)(\tan^2 x)$ ✓
Uses the identity $1 + \tan^2 x = 1/\cos^2 x$ ✓
Simplifies the integral to obtain $F(n)$ ✓
Uses the result from (a) ✓
Evaluates correctly ✓

(c) Using the result from (b), evaluate F(4). Show working. (3 marks)

$$F(4) = \frac{1}{2+1} - F(2)$$
$$= \frac{1}{3} - \left(\frac{1}{0+1} - F(0)\right)$$
$$= -\frac{2}{3} + \int_0^{\pi/4} dx$$
$$= \frac{\pi}{4} - \frac{2}{3}$$

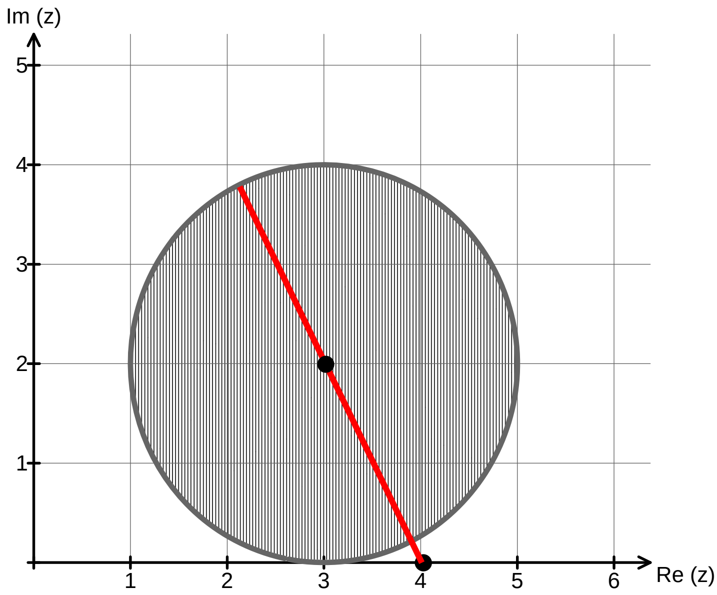
Expresses F(4) in terms of F(2) and F(2) in terms of F(0) with correct values of n ✓

Uses an integral to evaluate F(0) ✓

Simplifies correctly ✓

Question 18 **(8 marks)**

The figure below shows a shaded circle satisfied by a complex number z.



(a) Write an inequality that must be satisfied by z. (1 mark)

$|z-(3+2i)| \leq 2$

States the answer ✓

(b) Find the maximum exact value of $|z-4|$. (2 marks)

$$|z-4|_{max} = \sqrt{1^2+2^2}+2$$
$$= \sqrt{5}+2$$

Finds the distance between 3 + 2i and 4 ✓

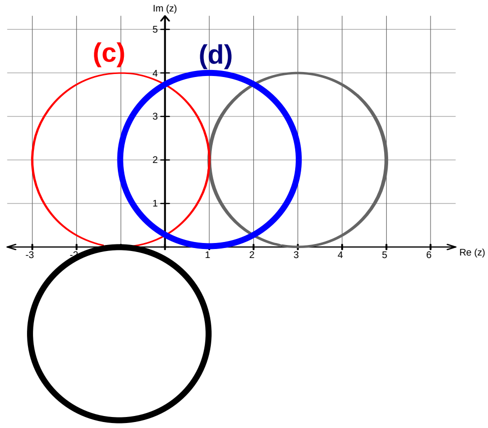
Adds 2 ✓

(c) Find the minimum value (in radians) of Arg (z – 4). (2 marks)

$$\text{min of } \arg(z-4)$$
$$= 2 \times \tan^{-1}\frac{1}{2}$$
$$= 0.93$$

Shows the new centre at -1 + 2i ✓

Uses correct trig ratio to find angle ✓



(d) Find the maximum value (in radians) of Arg (4 – $\bar{z}$). (3 marks)

$$z - 4 \rightarrow 4 - \bar{z}:$$
$$\text{reflection about the } x-\text{axis, rotation } 180°$$

$$\text{min of } \arg(4 - \bar{z})$$
$$= 2 \times \tan^{-1}\frac{2}{1}$$
$$= 2.21$$

Provides reason/description for 4 – $\bar{z}$ ✓

Shows the new centre at 1 + 2i ✓

Uses correct trig ratio to find angle ✓

Additional working space

Question number(s): ________________